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A unified approach for the solution of the Fokker–Planck equation

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Abstract. This paper explores the use of a discrete singular convolution algorithm as a unified approach for numerical integration of the Fokker–Planck equation. The unified features of the discrete singular convolution algorithm are discussed. It is demonstrated that different implementations of the present algorithm, such as global, local, Galerkin, collocation and finite difference, can be deduced from a single starting point. Three benchmark stochastic systems, the repulsive Wong process, the Black–Scholes equation and a genuine nonlinear model, are employed to illustrate the robustness and to test the accuracy of the present approach for the solution of the Fokker–Planck equation via a time-dependent method. An additional example, the incompressible Euler equation, is used to further validate the present approach for more difficult problems. Numerical results indicate that the present unified approach is robust and accurate for solving the Fokker–Planck equation.

1. Introduction

Much research has been done in the exploration of accurate and stable computational methods for the numerical solution of the Fokker–Planck equation [1–25]. A detailed comparison of several different approaches was given by Park and Petrosian [26] (see [25] for a literature review). In fact, the solution of the Fokker–Planck equation, in particular the nonlinear form of this equation, is still a non-trivial problem. In a somewhat broader sense, finding numerical solutions for partial differential equations (PDEs) is still a challenge owing to the presence of possible singularities and/or homoclinic manifolds that induce sharp transitions in the solutions [27]. These phenomena can be observed in many real systems such as black holes in astronomy, shock waves in compressible fluid flow, vortex sheets in incompressible flows associated with a high Reynolds number and burst events in the turbulent boundary layer. The difficulties associated with these phenomena can often be characterized by sharp changes occurring in a very small spatial region which can strongly influence the global properties of the system. The presence of these phenomena can be extremely sensitive to numerical algorithms and can easily lead to numerically induced spatial and/or temporal chaos [28]. At present, there are two major classes of numerical methods for solving PDEs, namely, global methods and local methods. In global methods, unknown functions and their derivatives are expanded in terms of a finite basis set with each element having a global support. The expansion coefficients are often determined by the method of tau, Galerkin, collocation or others. In the tau method, the residual for a truncated expansion is required to be orthogonal to a subset of basis functions used in the expansion, which, together with the boundary conditions, determines the expansion coefficients. In the global Galerkin method, a new set of basis functions is constructed by the superposition of the original basis functions. The

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requirement of the residual to be orthogonal to the new set of basis functions, together with the boundary conditions, determines the expansion coefficients. In the global collocation approach, the residual vanishes at a subset of nodal points of the highest-order basis function used in the expansion. The global collocation is also called the pseudospectral method. The three most important local approaches are the finite-difference, finite-volume and finite-element methods. In finite-difference methods, the solution is interpolated in terms of a set of grid values; the spatial derivatives are usually approximated by algebraic expressions involving nearest-neighbour grid points. In finite-volume approaches, the emphasis is on a set of integro-differential equations and their associated surface and volume integrations. The values on the boundary of each 'numerical molecule' are usually interpolated by low-order schemes. The spatial derivatives are approximated in the same way as those used in the finite-difference methods. Finite-element methods form one of the most versatile classes of numerical methods. Depending on the system under study, finite-element methods can be formulated either in terms of the method of weight residuals or in terms of variational principles. Usually, PDEs are integrated by using a set of trial functions, each with a small region of support. The solution is represented by linear superpositions of these trial functions.

Global methods are highly localized in their spectral space, but are unlocalized in the coordinate space. In contrast, local methods have high spatial localization, but are delocalized in their spectral space. In general, global methods are much more accurate than local methods, while the major advantage of local methods is their flexibility in handling complex geometries and boundary conditions. Moreover, the use of global methods is usually restricted to structured grids, whereas, local methods can be implemented on block-structured grids and even unstructured grids.

There were hectic debates among the numerical computation communities over the advantages and disadvantages of various numerical methods over the past few decades. These debates stimulated the development of powerful numerical methods for a wide variety of science and engineering applications. Such developments have, in association with the availability of inexpensive high-performance computers, led to the establishment of numerical simulations as an alternative approach for research and applications. The connection of various numerical methods has always been an important research topic. Finlayson discussed the relation between the Galerkin and the Ritz variational principle [29]. Canuto *et al* rearranged their spectral basis functions so that some global collocation methods could be regarded as a special case of certain global Galerkin methods [30]. Fornberg addressed the common feature between pseudospectral methods and high-order finite-difference methods [31]. The connection between global and local methods can also be realized within the framework of the method of weighted residuals by choosing trial functions of either piecewise Lagrange polynomials or global Lagrange polynomials. The connection of the finite-element, finite-difference and finite-volume methods is now well understood [32]. However, to the best of our knowledge, none has reported a unified scheme for the discussion of all of the above-mentioned methods.

In a previous work [25], we proposed a discrete singular convolution (DSC) algorithm and demonstrated its use for the numerical solution of the Fokker–Planck equation via eigenfunction expansions. The DSC algorithm was shown to be a potential numerical approach for the Hilbert transform, Abel transform, Radon transform and delta transform. Three standard problems, the Lorentz Fokker–Planck equation, the bistable model and the Henon–Heiles system, were utilized to test the accuracy, reliability and speed of convergence of the DSC eigenfunction approach. All results were in excellent agreement with those of previous methods in the field. Recently, the DSC algorithm has been successfully tested for integrating the sine–Gordon

equation with initial values close to homoclinic orbits [33], which is extremely difficult to compute because of the possible presence of numerical chaos [28]. Excellent results are obtained by solving the Navier–Stokes equation and for engineering structural analysis [34]. The purpose of the present paper is twofold. First, we study the unified features of the DSC algorithm for treating partial differential equations. This is accomplished by focusing on the DSC kernels of the delta type and their approximations. Second, we explore the use of the DSC as a unified approach for solving the Fokker–Planck equation via direct explicit time propagations. The eigenfunction expansion approach provides a Schrödinger-equation-type picture for understanding the Fokker–Planck equation. However, its use is restricted to a certain class of Fokker–Planck operators (essentially for the Fokker–Planck operators their equivalent Schrödinger potentials are bounded from below). The present direct approach is applicable to a wider class of problems. These two DSC-based approaches have the same level of accuracy as the numerical solution of the Fokker–Planck equation. They are complementary to each other for solving a wide variety of Fokker–Planck systems arising from practical situations.

This paper is organized as follows. The unified features of the DSC algorithm are discussed in section 2. We demonstrate that, the present DSC algorithm provides a unified framework for solving the Fokker–Planck equation, and partial differential equations in general. In particular, we show that various different implementations of the DSC algorithm, such as global, local, Galerkin, collocation and finite difference, can be deduced from a single starting point. The application of the present DSC approach to the solution of the Fokker–Planck equation and Euler equation is presented in section 3. We use four examples to illustrate the present approach. The first example is the repulsive Wong process which is useful for testing the ability to handle monomodality–bimodality transition. The second example is the Black–Scholes equation for option derivatives. This is an interesting stochastic model for option pricing in financial markets. The third case treated is a nonlinear stochastic model which has a certain connection to a mean-field model for self-organization processes in biological systems such as muscle contraction. Notably, all of these problems are treated by an explicit time-propagation approach in contrast to the eigenfunction expansion used in our previous work [25]. Since the above-mentioned examples are of strong parabolic type, we consider an additional problem, the incompressible Euler equation, to further validate the DSC approach for more difficult problems. The incompressible Euler equation is chosen because its equations for the velocity vector and pressure field are of strong hyperbolic and elliptic type, respectively. Thus, this last example is complimentary to three other examples from the point of view of numerical analysis. This paper ends with a discussion.

2. Properties of the discrete singular convolution

This section presents the properties of the DSC algorithm for solving differential equations. The first subsection addresses the unified features of the DSC algorithm in the line of the method of weighted residuals. Relevant properties of DSC trial functions are discussed in the second subsection.

2.1. Unified features

Without loss of generality, it is assumed that at a fixed time, a stochastic process is governed by a differential equation. To solve the differential equation, one can start either by approximating the original differential operator or by approximating the actual solution of the differential equation while maintaining the original differential operator. The latter is accomplished by

explicitly defining a functional form for approximations. Let us assume that the differential equation has the form

$$\mathcal{L}u(x) = f(x) \qquad x \in \Omega \tag{1}$$

where $\mathcal{L}$ is a linear operator and $u(x)$ is the unknown solution of interest. Here $f(x)$ is a known force term and Ω denotes the domain over which the differential equation applies.

The approximate solution is sought from a finite set of N DSC trial functions of a given *resolution* α, denoted by $S^{N,M}_{\alpha,\sigma}$ where M is the half-width of the support of each element. Here σ is a *regularization* parameter for improving the regularity of the set. The regularization-free case is easily obtained by setting $\sigma \to \infty$. Elements of the set $S^{N,M}_{\alpha,\sigma}$ can be given explicitly by $\{\phi^M_{\alpha,\sigma;1}, \phi^M_{\alpha,\sigma;2}, \ldots, \phi^M_{\alpha,\sigma;N}\}$. For a given computational domain, the resolution parameter α is determined by N.

An important property of the DSC trial functions $\{\phi^M_{\alpha,\sigma;k}\}$ is that when the trial function is free of regularization, each member of the set is a *reproducing kernel* at highest resolution

$$\lim_{\alpha\to\infty} \langle \phi^M_{\alpha,\sigma;k}, \eta\rangle = \eta(x_k) \tag{2}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the standard inner product. In fact, if an appropriate basis is used for ϕ and the limit on σ is taken, ϕ of each resolution can be a reproducing kernel for L^2 functions bandlimited in an appropriate sense. In general, we require the *low-pass filter property* that for given $\alpha \neq 0$, $\sigma \neq 0$ and $M \gg 0$

$$\langle \phi^M_{\alpha,\sigma;k}, \eta\rangle \approx \eta(x_k). \tag{3}$$

This converges uniformly when the resolution is refined, e.g. $\alpha \to \infty$. Many examples of such DSC trial functions are given in [25, 35]. Further discussion on these functions is given in the next subsection. Equations (2) and (3) are special requirements satisfied by the DSC kernels of delta type [25].

In the present DSC approach, an approximation to the function of interest $u(x)$ can be expressed as a linear combination

$$U^{N,M}_{\alpha,\sigma}(x) = \sum_{k=1}^{N} U_{\alpha,\sigma;k}\, \phi^M_{\alpha,\sigma;k}(x) \tag{4}$$

where x is an independent variable and $U_{\alpha,\sigma;k}$ is a DSC approximation to the solution required at point x_k. This structure is due to the DSC trial function property (3) and it dramatically simplifies the solution procedure in practical computations.

In this formulation, we choose the set $S^{N,M}_{\alpha,\sigma}$ *a priori*, and then determine the coefficients $\{U_{\alpha,\sigma;k}\}$ so that $U^{N,M}_{\alpha,\sigma}(x)$ is a good approximation to $u(x)$. To determine $U_{\alpha,\sigma;k}$, we minimize the amount by which $U^{N,M}_{\alpha,\sigma}(x)$ fails to satisfy the original governing equation (1). A measure of this failure can be defined as

$$R^{N,M}_{\alpha,\sigma}(x) \equiv \mathcal{L}U^{N,M}_{\alpha,\sigma}(x) - f(x) \tag{5}$$

where $R^{N,M}_{\alpha,\sigma}(x)$ is the residual for a particular choice of resolution, regularization and half-width of the support. Note that equation (5) is similar to the usual statement in the method of weighted residuals. However, the approximation $U^{N,M}_{\alpha,\sigma}(x)$ is constructed by using the DSC trial functions, $\phi^M_{\alpha,\sigma;k}(x)$, in the present treatment. Let equation (1) and its associated boundary conditions be well posed, then there exists a unique solution $u(x)$ which generally resides in an infinite-dimensional space. Since the DSC approximation $U^{N,M}_{\alpha,\sigma}$ is constructed from a finite-dimensional set, it is generally the case that $U^{N,M}_{\alpha,\sigma}(x) \neq u(x)$ and therefore $R^{N,M}_{\alpha,\sigma}(x) \neq 0$.

Galerkin. We seek to optimize $R^{N,M}_{\alpha,\sigma}(x)$ by forcing it to zero in a *weighted average sense* over the domain Ω. A convenient starting point is

$$\int_\Omega R^{N,M}_{\alpha,\sigma}(x)\,\phi^{M'}_{\alpha',\sigma';l}(x)\,\mathrm{d}x = 0 \qquad \phi^{M'}_{\alpha',\sigma';l}(x) \in S^{N',M'}_{\alpha',\sigma'} \tag{6}$$

where the weight set $S^{N',M'}_{\alpha',\sigma'}$ can be simply chosen to be identical to the DSC trial function set $S^{N,M}_{\alpha,\sigma}$. We refer to equation (6) as a DSC-Galerkin statement.

Collocation. First, we note that in view of equation (2), the present DSC-Galerkin statement reduces to a collocation one at the limit of α'

$$\lim_{\alpha'\to\infty}\int_\Omega R^{N,M}_{\alpha,\sigma}(x)\,\phi^{M'}_{\alpha',\sigma';l}(x)\,\mathrm{d}x = R^{N,M}_{\alpha,\sigma}(x_l) = 0 \tag{7}$$

where $\{x_l\}$ is the set of collocation points. However, in digital computations, we cannot take the above limits. It follows from the low-pass filter property of the DSC trial functions, equation (3), that

$$\int_\Omega R^{N,M}_{\alpha,\sigma}(x)\,\phi^{M'}_{\alpha',\sigma';l}(x)\,\mathrm{d}x \approx R^{N,M}_{\alpha,\sigma}(x_l) \approx 0. \tag{8}$$

It can be proven that for an appropriate choice of $S^{N',M'}_{\alpha',\sigma'}$, the first approximation of equation (8) converges uniformly. The difference between the true DSC-collocation,

$$\lim_{\alpha'\to\infty} R^{N,M}_{\alpha,\sigma}(x_l) = 0 \tag{9}$$

and the *Galerkin-induced collocation*, (8), diminishes to zero for appropriate DSC trial functions.

Global and local. Global approximations to a function and its derivatives are realized typically by a set of truncated $L^2(a,b)$ function expansions. It is called global because the values of a function and its derivatives at a particular point x_i in the coordinate space involve the *full* set of grid points in a computational domain Ω. Whereas a local method does so by requiring only a few nearest-neighbour points. In the present DSC approach, since the choices of M and/or M' are independent of N, one can choose M and/or M' so that a function and its derivatives at a particular point x_l are approximated either by the full set of grid points in the computational domain Ω or just by a few nearest-neighbour grid points. In fact, this freedom for the selection of M endows the DSC algorithm with *controllable accuracy* for solving differential equations and the flexibility for handling complex geometries.

Finite difference. In the finite-difference method, the differential operator is approximated by difference operations. In the present approach, the DSC-collocation expression of equation (8) is equivalent to a $(2M+1)$- (or $2M$-) term finite-difference method. This follows from the fact that the DSC approximation to the nth-order derivative of a function can be rewritten as

$$\left.\frac{\mathrm{d}^q u}{\mathrm{d}x^q}\right|_{x=x_k} \approx \sum_{l=k-M}^{k+M} c^q_{kl,M}\,u(x_l) \tag{10}$$

where $c^q_{kl,M}$ are a set of DSC weights for the finite-difference approximation and are given by

$$c^q_{kl,M} = \left.\frac{\mathrm{d}^q}{\mathrm{d}x^q}\phi^M_{\alpha,\sigma;l}(x)\right|_{x=x_k}. \tag{11}$$

Obviously, for each different choice of $\phi^M_{\alpha,\sigma}$, we have a different DSC finite-difference approximation. Hence, the present DSC approach is a generalized finite-difference method. This DSC finite difference was tested in previous studies [35]. When $M = 1$, the DSC finite-difference approximation reaches its low-order limit and the resulting matrix is tridiagonal. In this case, the present DSC weights $c^q_{kl,M}$ can always be made exactly the same as those of the second-order central-difference scheme (i.e. $\frac{1}{2\Delta}, 0, -\frac{1}{2\Delta}$ for the first-order derivative and $\frac{1}{\Delta^2}, -\frac{2}{\Delta^2}, \frac{1}{\Delta^2}$ for the second-order derivative, where Δ is the grid spacing) of the standard finite-difference method by choosing the parameter σ appropriately. However, even in this case, the DSC finite-difference approximation does not have to be the same as the standard finite-difference scheme and can be optimized in a practical application by varying σ.

2.2. DSC trial functions

There are many DSC trial functions that satisfy equation (3). The requirement of equation (3) can be regarded as an *approximate reproducing kernel* or quasi-reproducing kernel. The reason for using an approximate reproducing kernel can be understood from the following analysis of Shannon's kernel $\frac{\sin(\alpha x)}{\pi x}$. Shannon's kernel is a delta sequence

$$\lim_{\alpha\to\infty} \frac{\sin(\alpha x)}{\pi x} = \delta(x) \tag{12}$$

where $\delta(x)$ is the delta distribution which can be regarded as a *universal reproducing kernel* because its Fourier transform is unity. However, such a universal reproducing kernel cannot be used directly in digital computations because it is a distribution (precisely, it belongs to Sobolev space of order -1, H^{-1}) and it does not have a value anywhere in the coordinate space. Therefore, in a certain sense, constructing a reproducing kernel in an appropriate $L^2(a, b)$ space is equivalent to finding a sequence of approximations of the delta distribution in the $L^2(a, b)$. In fact, Shannon's kernel is an element of the Paley–Wiener reproducing kernel Hilbert space B^2_π,

$$f(x) = \int_{-\infty}^{\infty} f(y) \frac{\sin \pi(x - y)}{\pi(x - y)} \,\mathrm{d}y \qquad \forall f \in B^2_\pi \tag{13}$$

where $\forall f \in B^2_\pi$ indicates that, in its Fourier representation, the L^2 function f vanishes outside the interval $[-\pi, \pi]$. What is important for digital computations is the fact that the Paley–Wiener reproducing kernel Hilbert space has a *sampling basis* $S_k(x)$

$$S_k(x) = \frac{\sin \pi(x - y_k)}{\pi(x - y_k)} \qquad y_k = k \qquad \forall k \in \mathcal{Z} \tag{14}$$

where the symbol $\mathcal{Z}$ denotes the set of all integers. Expression (14) provides a discrete representation of every (continuous) function in B^2_π

$$f(x) = \sum_{k\in\mathcal{Z}} f(y_k) S_k(x) \qquad \forall f \in B^2_\pi. \tag{15}$$

This is Shannon's sampling theorem and is particularly important in information and sampling theory. Note that Shannon's kernel is obviously interpolative on $\mathcal{Z}$,

$$S_n(x_m) = \delta_{n,m} \tag{16}$$

where $\delta_{n,m}$ is the Kronecker delta function. Computationally, being interpolative is of particular importance for numerical accuracy and simplicity.

In wavelet analysis, $\frac{\sin(\pi x)}{\pi x}$ is Shannon's wavelet scaling function and its Fourier transform is a characteristic function, i.e. it is an unsmoothed, *ideal* low-pass filter. In physical language,

it is a projection to the frequency subband $[-\pi, \pi]$. By the Heisenberg uncertainty principle, such a (sharp) projection must be an infinite impulse response (IIR) filter. The usefulness of such a filter is limited because it is delocalized in the coordinate space and requires infinitely many sampling data. In practical computations, a truncation is required, which leads to a large truncation error and even worse, numerical instability. To improve the smoothness and regularity of Shannon's kernel, we introduce a regularization

$$\Phi_\sigma(x) = \frac{\sin(\pi x)}{\pi x} R_\sigma(x) \qquad (\sigma > 0) \tag{17}$$

where R_σ is a *regularizer* which has properties

$$\lim_{\sigma\to\infty} R_\sigma(x) = 1 \tag{18}$$

and

$$R_\sigma(0) = 1. \tag{19}$$

Here equation (18) is a general condition that a regularizer must satisfy, while equation (19) is specifically for a *delta regularizer*, which is used in regularizing a delta kernel. Various delta regularizers can be used for numerical computations. An excellent one is the Gaussian

$$R_\sigma(x) = \exp\left[-\frac{x^2}{2\sigma^2}\right]. \tag{20}$$

An immediate benefit of the regularized Shannon's kernel, equation (17), is that its Fourier transform is infinitely differentiable because the Gaussian is an element of the Schwartz class functions. Qualitatively, all kernels of the Dirichlet type oscillate in the coordinate representation. Specifically, Shannon's kernel has a long tail which is proportional to $\frac{1}{x}$, whereas, the regularized kernels decay exponentially fast, especially when σ is very small. In the Fourier representation, regularized Shannon kernels have an 'optimal' shape in their frequency responses. Of course, they all reduce to Shannon's low-pass filter in the limit

$$\lim_{\sigma\to\infty} \Phi_\sigma(x) = \lim_{\sigma\to\infty} \frac{\sin \pi x}{\pi x} \exp\left[-\frac{x^2}{2\sigma^2}\right] = \frac{\sin \pi x}{\pi x}. \tag{21}$$

Quantitatively, one can examine the normalization of $\Phi_\sigma(x)$

$$\begin{aligned}
\int \Phi_\sigma(x)\,\mathrm{d}x &= \hat{\Phi}_\sigma(0) \\
&= \sqrt{2\pi}\sigma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{\pi\sigma}{\sqrt{2}}\right)^{2k} \qquad (22) \\
&= \operatorname{erf}\left(\frac{\pi\sigma}{\sqrt{2}}\right) \\
&= 1 - \sqrt{\frac{2}{\pi}}\frac{1}{\sigma} \exp\left[-\tfrac{1}{2}\sigma^2\pi^2\right] \int_0^\infty \exp\left[-\frac{t^2}{2\sigma^2} - \pi t\right] \mathrm{d}t \\
&= 1 - \operatorname{erfc}\left(\frac{\pi\sigma}{\sqrt{2}}\right) \qquad (23) \\
&\neq 1 \qquad (24)
\end{aligned}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t$$

is the error function and erfc(z) is the complementary error function. Note that for a given $\sigma > 0$, erfc($\pi\sigma/\sqrt{2}$) is positive definite. Thus, $\hat{\Phi}_\sigma(0)$ is always less than unity except at the limit of $\sigma \to \infty$. Therefore, $\Phi_\sigma(x)$ is no longer a reproducing kernel. However, we argue that $\Phi_\sigma(x)$ is an approximate reproducing kernel because when we choose $\sigma \gg \sqrt{2}/\pi$, which is the case in many practical applications, the residue term, erfc($\pi\sigma/\sqrt{2}$), approaches zero very quickly. As a result, $\hat{\Phi}_\sigma(0)$ is extremely close to unity. As trial functions, regularized Shannon kernels do not form a sampling basis. They are no longer orthogonal in general. However, they just *slightly* miss the orthogonality and the requirement of a basis.

For numerical computations, it turns out that the approximate reproducing kernel has much fewer truncation errors for interpolation and numerical differentiations. Qian and the present author [36] have recently given the following theorem for truncation errors.

Theorem. *Let f be a function $f \in L^2(R) \cap C^s(R)$, bandlimited to B, ($B < \frac{\pi}{\Delta}$, Δ is the grid spacing). For a fixed $t \in R$ and $\sigma > 0$, denote $g(x) = f(x)H_k((t-x)/\sqrt{2}\sigma)$, where $H_k(x)$ is the kth-order Hermite polynomial. If $g(x)$ satisfies*

$$g'(x) \leqslant g(x)\frac{(x-t)}{\sigma^2} \tag{25}$$

for $x \geqslant t + (M_1 - 1)\Delta$, and

$$g'(x) \geqslant g(x)\frac{(x-t)}{\sigma^2} \tag{26}$$

for $x \leqslant t - M_2\Delta$, where $M_1, M_2 \in \mathcal{N}$, then for any $s \in \mathcal{Z}^+$

$$\begin{aligned}
&\left\| f^{(s)}(t) - \sum_{n=\lceil t/\Delta\rceil - M_2}^{\lceil t/\Delta\rceil + M_1} f(n\Delta)\left[\frac{\sin[(\pi/\Delta)(t-n\Delta)]}{(\pi/\Delta)(t-n\Delta)}\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]^{(s)}\right\|_{L^2(R)} \\
&\quad\leqslant \sqrt{3}\Bigg[\frac{\|f^{(s)}(t)\|_{L^2(R)}}{2\pi\sigma(\pi/\Delta - B)\exp\left[\frac{1}{2}\left(\sigma^2(\pi/\Delta - B)^2\right)\right]} \\
&\qquad + \|f(t)\|_{L^2(R)}\left[\sum_{i+j+k=s}\frac{s!\pi^{i-1}H_k(-M_1\Delta/\sqrt{2}\sigma)}{i!k!\Delta^{i-1}(\sqrt{2}\sigma)^k((M_1-1)\Delta)^{j+1}}\right]\exp\left(-\frac{(M_1\Delta)^2}{2\sigma^2}\right) \\
&\qquad + \|f(t)\|_{L^2(R)}\left[\sum_{i+j+k=s}\frac{s!\pi^{i-1}H_k(-M_2\Delta/\sqrt{2}\sigma)}{i!k!\Delta^{i-1}(\sqrt{2}\sigma)^k(M_2\Delta)^{j+1}}\right]\exp\left(-\frac{(M_2\Delta)^2}{2\sigma^2}\right)\Bigg]
\end{aligned} \tag{27}$$

where the superscript s denotes the sth-order derivative.

The proof and detailed discussion (including a comparison with the truncation errors of Shannon's sampling theorem) are given in [36] and are beyond the scope of this paper.

This theorem provides a guide to the choice of M, σ and Δ. For example, in the case of interpolation ($s = 0$), if the L_2 norm error is set to $10^{-\eta}$ ($\eta > 0$), the following relations can be deduced from equation (27):

$$r(\pi - B\Delta) > \sqrt{4.61\eta} \tag{28}$$

and

$$\frac{M}{r} > \sqrt{4.61\eta} \tag{29}$$

where $r = \sigma/\Delta$. (The choice of σ is always proportional to Δ so that the width of the Gaussian envelope varies with the central frequency.) The first inequality states that for a

given grid size Δ, a large r is required for approximating the high-frequency component of an L^2 function. The second inequality indicates that if one chooses the ratio $r = 3$, then the half bandwidth $M \sim 30$ can be used to ensure the highest accuracy in a double-precision computation ($\eta = 15$). However, for lower accuracy requirement, a much smaller half bandwidth can be used. In general, the value of r is proportional to M. The use of M values is determined by the accuracy requirement. This theoretical estimation is in excellent agreement with a previous numerical test [35].

3. Illustrative calculations

In this section, we illustrate the use of the present approach for solving the Fokker–Planck equation and the incompressible Euler equation. Many DSC kernels are discussed in the previous work [25, 35] and they can be used as the DSC trial functions. For simplicity, we focus on three DSC kernels, a regularized Shannon's kernel (RSK),

$$\phi^M_{(\pi/\Delta),\sigma;k}(x) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right] \tag{30}$$

a regularized Dirichlet kernel (RDK),

$$\phi^M_{(\pi/\Delta),\sigma;k}(x) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(2m + 1)\sin[(\pi/\Delta)(x - x_k)/(2m + 1)]} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right] \tag{31}$$

and a regularized Lagrange kernel (RLK)

$$\phi^M_{(\pi/\Delta),\sigma;k}(x) = \prod_{i\neq k}^{2m} \frac{x - x_i}{x_k - x_i} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right] \tag{32}$$

for our numerical test. Note that the resolution is given by $\alpha = \frac{\pi}{\Delta}$, which is the frequency bound in the Fourier representation. The goal of this section is to test the present method for the solutions of the Fokker–Planck equation via time propagation and the incompressible Euler equation. For the numerical solution of the Fokker–Planck equation, we choose $\sigma = 3.8\Delta$ for the RSK and RDK, $\sigma = 2.8\Delta$ for the RLK, where π/Δ is the resolution. In fact, a wide range of σ values can be used to deliver excellent results. The half bandwidth, M, can be chosen to interplay between the local limit and the global limit and is set to 40 in all calculations. Finally, m controls the order of the regularized Dirichlet and Lagrange kernels and is set to 40 in all calculations (note that the selection of m is independent of the grid used in the computation). It is noted that all of the above-mentioned DSC trial functions are of Schwartz class and are capable of auto-regularizing when used as integral kernels. The fourth-order explicit Runge–Kutta scheme is used for time discretization. Details of these computations are described in the first three subsections. For treating the incompressible Euler equation, many other DSC parameters are tested as indicated in the last subsection. Double precision is used in all calculations.

3.1. The repulsive Wong process

One of important stochastic systems is the repulsive Wong process [20, 37–39], given by

$$\mathrm{d}x = 2\gamma \tanh(x)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}F_t \tag{33}$$

where $\mathrm{d}F_t$ is the Gaussian white noise which has the standard statistical properties

$$\langle \mathrm{d}F_t \rangle = 0 \tag{34}$$

and

$$\langle \mathrm{d}F_t, \mathrm{d}F_\tau \rangle = \delta(|t-\tau|). \tag{35}$$

The repulsive Wong process is Markovian due to the deriving Gaussian white noise term. Its transition probability density is governed by the Fokker–Planck equation of the form [37–39]

$$\frac{\partial f(x,t)}{\partial t} = -2\gamma \frac{\partial[\tanh(x) f(x,t)]}{\partial x} + \frac{\partial^2 f(x,t)}{\partial x^2} \tag{36}$$

with the usual initial condition

$$f(x,0) = \delta(x - x_0) \tag{37}$$

and the normalization

$$\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}t = 1. \tag{38}$$

For $\gamma = 1$, the solution [38, 39] of the Fokker–Planck equation (36) is analytically given by (for $x_0 = 0$)

$$f(x,t) = \frac{1}{2\sqrt{4\pi t}} \left[\exp\left[-\frac{(x-x_-)^2}{4t} \right] + \exp\left[-\frac{(x-x_+)^2}{4t} \right] \right] \tag{39}$$

where $x_\pm = \pm 2t$ are centres of two moving Gaussians. Here the superposition of two Gaussians gives rise to a monomodality–bimodality transition as time increases. The Wong process is useful for illustrating the connection between stochastic processes and quantum measurements. It is also useful for distinguishing spectrum differences between the Master equation and its Fokker–Planck equation approximations.

The accurate simulation of the Wong process is not a simple task because of the monomodality–bimodality transition. Two Gaussian peaks centred at $x_\pm = \pm 2t$ move apart as time increases. The computational domain has to be sufficiently large in order to avoid boundary reflection (otherwise, more complicated techniques, such as absorption boundaries, have to be implemented). In the present computations, the resolution is chosen as $\frac{\pi}{\Delta} = 10\pi$. The initial functions are approximated by a unit impulse function located at 0. Equation (36) is integrated up to 10 time units with a time increment of 0.001. The errors for a wide range of propagation times are listed in table 1 and are measured by error norms of L_2 and L_∞. It is seen that the present unified approach is extremely accurate and reaches the level of machine precision. All of the DSC kernels behave very similarly to each other and provide the same level of accuracy and speed of convergence. In fact, other DSC kernels, such as the regularized modified Dirichlet kernel, provide similar results. The results of the RSK and RDK are slightly more accurate than those of the RLK. It is evident that the present unified method, in association with the DSC trial functions, is capable of delivering extremely high accuracy and numerical stability for the Wong process. To the best of our knowledge, the DSC solution for this system is the best to date.

3.2. The Black–Scholes equation

The Fokker–Planck equation and stochastic analysis have interesting applications in mathematical modelling of financial market option pricing. Consider a writer of a European call option on a stock, he is exposed to the risk of unlimited liability if the stock price rises acutely above the strike price. To protect his short position in the option, he should consider purchasing a certain amount of stock so that the loss in the short position in the option is offset

Table 1. Errors for solving the repulsive Wong process.

	RSK		RDK		RLK	
Time	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.10	1.94(−09)	2.32(−09)	1.94(−09)	2.32(−09)	1.94(−09)	2.32(−09)
0.25	4.67(−11)	3.73(−11)	4.67(−11)	3.73(−11)	4.67(−11)	3.72(−11)
0.50	3.47(−12)	2.05(−12)	3.47(−12)	2.05(−12)	3.47(−12)	2.03(−12)
0.75	8.76(−13)	4.89(−13)	8.76(−13)	4.95(−13)	8.83(−13)	4.60(−13)
1.00	3.57(−13)	1.83(−13)	3.57(−13)	1.91(−13)	3.78(−13)	1.91(−13)
2.00	6.29(−14)	2.69(−14)	5.92(−14)	3.00(−14)	2.18(−13)	8.92(−14)
3.00	4.72(−14)	2.04(−14)	3.60(−14)	1.63(−14)	2.86(−13)	9.76(−14)
4.00	5.23(−14)	1.97(−14)	3.76(−14)	1.39(−14)	3.53(−13)	1.12(−13)
5.00	4.44(−14)	1.19(−14)	3.16(−14)	1.12(−14)	4.10(−13)	1.21(−13)
6.00	5.87(−14)	1.88(−14)	5.23(−14)	1.64(−14)	4.68(−13)	1.32(−13)
7.00	7.63(−14)	2.38(−14)	7.30(−14)	2.16(−14)	5.25(−13)	1.43(−13)
8.00	9.27(−14)	2.78(−14)	9.08(−14)	2.58(−14)	5.80(−13)	1.54(−13)
9.00	7.19(−14)	1.99(−14)	6.24(−14)	1.79(−14)	6.33(−13)	1.63(−13)
10.0	6.98(−14)	1.70(−14)	5.25(−14)	1.39(−14)	6.87(−13)	1.73(−13)

by the long position in the stock. In this way, he is adopting a hedging procedure. A hedge position combines an option with its underlying asset so as to achieve the goal that either the stock protects the option against loss or the option protects the stock against loss. This risk-monitoring strategy has been commonly used by practitioners in financial markets. The most well known stochastic model for the equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate is the Black–Scholes equation [40]

$$\frac{\partial c}{\partial t} = \frac{\nu^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc \tag{40}$$

where S is the asset price which undergoes geometric Brownian motion, $c(S, t)$ the call price, ν the volatility and r the constant riskless interest rate. Black–Scholes equation is a fundamental equation in finance and economics and is also an excellent example application of stochastic analysis. By a simple transformation

$$x = \ln S \tag{41}$$

and

$$f(x, t) = \mathrm{e}^{rt} c(x, t) \tag{42}$$

the Black–Scholes equation is transformed into the Fokker–Planck equation of the standard form

$$\frac{\partial f}{\partial t} = \left(r - \tfrac{1}{2}\nu^2\right) \frac{\partial f}{\partial x} + \frac{\nu^2}{2} \frac{\partial^2 f}{\partial x^2}. \tag{43}$$

The numerical simulation of the Black–Scholes equation and its generalized versions is an important issue in financial analysis and the computational finance community [41–44]. Essentially, all existing numerical methods are tested for potential usefulness in estimating the option derivatives because both computational accuracy and efficiency are very important to option modelling and risk estimation. In the present time-dependent approach, the resolution is set to $\frac{\pi}{\Delta} = 2\pi$ and the time increment is chosen as 0.01. For simplicity, we choose $\frac{1}{2}\nu^2 = 0.5$ and $r = 0.7$ in our calculations. We have chosen our initial distribution as a unit

Table 2. Errors for the numerical solution of the Black–Scholes equation.

	RSK		RDK		RLK	
Time	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	1.85(−03)	1.14(−03)	1.83(−03)	1.12(−03)	2.18(−03)	1.41(−03)
2	1.19(−04)	8.55(−05)	1.18(−04)	8.45(−05)	1.42(−04)	1.06(−04)
3	4.57(−06)	2.89(−06)	4.48(−06)	2.84(−06)	6.44(−06)	4.29(−06)
4	2.53(−07)	1.68(−07)	2.46(−07)	1.63(−07)	4.23(−07)	2.84(−07)
5	1.81(−08)	1.15(−08)	1.75(−08)	1.10(−08)	3.66(−08)	2.43(−08)
6	1.63(−09)	1.09(−09)	1.56(−09)	1.04(−09)	4.02(−09)	2.74(−09)
7	1.81(−10)	1.26(−10)	1.71(−10)	1.19(−10)	5.48(−10)	3.92(−10)
8	2.41(−11)	1.59(−11)	2.25(−11)	1.48(−11)	9.03(−11)	6.05(−11)
9	3.83(−12)	2.55(−12)	3.54(−12)	2.37(−12)	1.76(−11)	1.15(−11)
10	7.82(−13)	6.25(−13)	7.30(−13)	5.85(−13)	4.01(−12)	2.84(−12)
20	4.86(−14)	2.70(−14)	4.91(−14)	2.78(−14)	4.84(−14)	2.74(−14)

impulse function located at $x = 0$, which is a poor approximation to the true delta distribution. Obviously, had one started with a smooth initial function, or used a denser grid, one would have obtained much higher accuracy at earlier times as well. We have verified this computationally, but these results are not presented. Both L_2 and L_∞ error analyses are used to evaluate the quality of the DSC approach, the results of which are listed in table 2. To the best of our knowledge, the present time-dependent DSC approach provides the most accurate numerical results yet obtained for the Black–Scholes equation.

As in the first example, three DSC kernels provide extremely similar results in solving the Black–Scholes equation. This is not an isolated coincidence. In fact, we can come up with a number of other DSC kernels where all of their results are very similar to those of the present three kernels.

3.3. A nonlinear stochastic model

To illustrate the accuracy and robustness of the present approach further, we choose the following nonlinear stochastic model:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial[(\omega x + \theta\langle x(t)\rangle) f(x,t)]}{\partial x} + D\frac{\partial^2 f(x,t)}{\partial x^2} \tag{44}$$

where $\langle x(t)\rangle$ is the first moment of the distribution function

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} x f(x,t)\,\mathrm{d}x \tag{45}$$

and ω, θ and D are constant. The initial probability distribution is also given by

$$f(x,0) = \delta(x - x_0). \tag{46}$$

Equation (44) is a true nonlinear stochastic model since the instantaneous position average depends on the distribution function. This is one of few analytically solvable nonlinear systems which are very valuable for testing new numerical approaches. For example, Drozdov and Morillo have recently employed this system to test their K-point Stirling interpolation formula finite-difference method [23]. The exact solution to equation (44) is

$$f(x,t) = \frac{1}{\sqrt{2\pi\nu(t)}} \exp\left[-\frac{(x - \langle x(t)\rangle)^2}{2\nu(t)}\right] \tag{47}$$

Table 3. Errors for solving the nonlinear model.

	RSK		RDK		RLK	
Time	L_1	L_∞	L_1	L_∞	L_1	L_∞
1	4.13(−01)	3.09(−02)	6.46(−01)	4.71(−02)	4.67(−02)	3.98(−03)
2	4.66(−01)	3.68(−02)	7.41(−01)	5.86(−02)	2.48(−03)	2.27(−04)
3	2.03(−01)	1.68(−02)	3.26(−01)	2.68(−02)	7.24(−04)	6.51(−05)
4	4.88(−02)	4.92(−03)	7.97(−02)	8.04(−03)	1.23(−04)	1.31(−05)
5	7.41(−03)	7.84(−04)	1.22(−02)	1.29(−03)	1.68(−05)	1.82(−06)
6	1.02(−03)	1.12(−04)	1.68(−03)	1.84(−04)	2.26(−06)	2.48(−07)
7	1.38(−04)	1.52(−05)	2.28(−04)	2.50(−05)	3.06(−07)	3.36(−08)
8	1.87(−05)	2.05(−06)	3.09(−05)	3.38(−06)	4.14(−08)	4.54(−09)
9	2.53(−06)	2.77(−07)	4.17(−06)	4.58(−07)	5.59(−09)	6.13(−10)
10	3.42(−07)	3.75(−08)	5.65(−07)	6.19(−08)	7.56(−10)	8.29(−11)
20	3.64(−14)	3.11(−15)	4.75(−14)	4.88(−15)	1.00(−13)	1.51(−14)

where $\langle x(t)\rangle$ and $v(t)$ are given analytically by

$$\langle x(t)\rangle = x_0 \mathrm{e}^{-(\omega+\theta)t} \tag{48}$$

and

$$v(t) = \frac{D}{\omega}\left(1 - \mathrm{e}^{-2\omega t}\right) \tag{49}$$

respectively. Obviously, $v(t)$ is the theoretical value of the second moment $M_2(t)$

$$M_2(t) = \langle x^2(t)\rangle - \langle x(t)\rangle^2 \tag{50}$$

which can also be used as a measure of computational accuracy.

In the present computations, the resolution is chosen as $\frac{\pi}{\Delta} = \frac{239}{20}\pi$. The time increment is taken as $\Delta t = 0.005$. In this example, the errors are measured by error norms of L_1 and L_∞ from which all other error norms, such as the L_2 error norm, can be interpolated. The L_1 and L_∞ errors are listed in table 3, for $D = 0.1, \omega = 1, \theta = 2$ and $x_0 = 2.0422$. The initial accuracy of computations is hindered by the poor approximation of the impulse function to the Dirac delta function. However, the auto-regularization property of the Schwartz class trial functions enables the numerical integration to stabilize on a smooth solution and eventually reach the machine precision at a slightly later time.

3.4. The Euler equation

All cases considered in the last three subsections are of strong parabolic type with a solution which becomes more and more flat and smooth as time increases. In this subsection, we consider an additional problem, the incompressible Euler equation, to confirm that the results obtained for the Fokker–Planck equation are not due to the parabolic nature. We also use this example to demonstrate the interconnection between the collocation and the finite difference, and between the local and the global methods. It is hoped that this additional example helps to build confidence in using the DSC approach for treating more difficult problems. Conceptually and numerically, it is convenient to describe the incompressible Euler equation from the point of view of the incompressible Navier–Stokes equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{v} \tag{51}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{52}$$

where $\boldsymbol{v}$ is the velocity field vector, p is the pressure field and Re is the Reynolds number. The Euler equation is attained by setting $Re = \infty$. Finding a general solution to the Euler equation is not an easy job. In the present study, we consider a solution domain of $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. Under such a constraint, the Navier–Stokes equation (51) has an exact solution

$$\begin{aligned} u(x, y, t) &= -\cos(x)\sin(y)\,\mathrm{e}^{-2t/Re} \\ v(x, y, t) &= \sin(x)\cos(y)\,\mathrm{e}^{-2t/Re} \\ p(x, y, t) &= -\tfrac{1}{4}[\cos(2x) + \cos(2y)]\,\mathrm{e}^{-4t/Re} \end{aligned} \tag{53}$$

where (u, v) are the velocity components in the x- and y-directions, respectively. Note that, for the Euler equation, the solution (53) does not decay with time.

In the present study, we use a standard approach for treating the incompressible Navier–Stokes equation, i.e. deriving a Poisson equation for the pressure from the incompressible condition. The velocity fields are iterated by using the implicit Euler scheme. At time t_{n+1}, there are two coupled equations for the velocity field

$$\left(\frac{1}{Re}\nabla^2 - \frac{1}{\Delta t}\right) u^{n+1} = p_x^{n+1/2} + S_x^n \tag{54}$$

$$\left(\frac{1}{Re}\nabla^2 - \frac{1}{\Delta t}\right) v^{n+1} = p_y^{n+1/2} + S_y^n \tag{55}$$

and a Poisson equation for the pressure

$$\nabla^2 p^{n+1/2} = S_p^n. \tag{56}$$

Here S_x^n, S_y^n and S_p^n are given by

$$\begin{aligned} S_x^n &= -\frac{u^n}{\Delta t} + (u^n u_x^n + v^n u_y^n) \\ S_y^n &= -\frac{v^n}{\Delta t} + (u^n v_x^n + v^n v_y^n) \\ S_p^n &= \frac{1}{\Delta t}(u_x^n + v_y^n) - (u_x^n)^2 - (v_y^n)^2 - 2u_y^n v_x^n. \end{aligned} \tag{57}$$

At each time t_{n+1}, the pressure field $p^{n+1/2}$ is solved according to equation (56) from the known velocity field vector (u^n, v^n). The velocity field vector (u^{n+1}, v^{n+1}) is then updated according to equations (54) and (55). These linear algebraic equations are solved by using a standard (LU decomposition) solver.

The derivatives in equations (54)–(56) are computed by using the generalized finite-difference scheme, equation (10), and the required finite difference weights are given by equation (11). The involved trial functions, $\phi_{\alpha,\sigma}^M$, are given by the RSK (equation (30)), the RDK (equation (31)) and the RLK (equation (32)). Here $m = 40$ is used for both RDK and RLK. We choose a small time increment ($\Delta t = 0.001$) so that the main error is caused by the spatial discretization. The number of grid points in each dimension is chosen as $N = 4, 8, 16$ and 32 in various test calculations. The α value is specified as

$$\alpha = \frac{\pi}{\Delta} = \frac{\pi}{2\pi/(N-1)} = \frac{N-1}{2}.$$

For a given N, the matrix half bandwidth, M, can be chosen as $M \leqslant N$. In particular, if $M = N$, the approach has a global (full) computational matrix. For all $M < N$, the matrix is banded. In the present DSC approach, the connection between the global and the local can

Table 4. Errors for solving the Euler equation.

			RSK		RDK		RLK	
N	M	Time	L_1	L_∞	L_1	L_∞	L_1	L_∞
4	1	0.5	3.16(−2)	6.12(−2)	3.15(−2)	6.12(−2)	3.09(−2)	6.01(−2)
		1.0	3.10(−2)	6.02(−3)	3.09(−2)	6.01(−3)	3.03(−2)	5.88(−3)
		1.5	3.06(−2)	5.90(−3)	3.05(−2)	5.89(−3)	2.99(−2)	5.73(−3)
		2.0	3.04(−2)	5.77(−3)	3.04(−2)	5.76(−3)	2.99(−2)	5.57(−3)
	2	0.5	1.27(−2)	2.48(−3)	1.27(−2)	2.48(−3)	1.29(−2)	2.44(−3)
		1.0	1.31(−2)	2.66(−3)	1.31(−2)	2.66(−3)	1.29(−2)	2.54(−3)
		1.5	1.37(−2)	2.86(−3)	1.37(−2)	2.87(−3)	1.32(−2)	2.66(−3)
		2.0	1.44(−2)	3.08(−3)	1.45(−2)	3.08(−3)	1.35(−2)	2.78(−3)
	4	0.5	9.33(−3)	1.70(−3)	9.32(−3)	1.69(−3)	9.34(−3)	1.70(−3)
		1.0	9.43(−3)	1.79(−3)	9.42(−3)	1.79(−3)	9.44(−3)	1.79(−3)
		1.5	9.62(−3)	1.88(−3)	9.61(−3)	1.88(−3)	9.64(−3)	1.88(−3)
		2.0	9.92(−3)	1.96(−3)	9.91(−3)	1.95(−3)	9.93(−3)	1.96(−3)
8	8	0.5	1.30(−4)	4.26(−5)	1.33(−4)	4.36(−5)	1.24(−4)	3.40(−5)
		1.0	1.52(−4)	5.13(−5)	1.54(−4)	5.25(−5)	1.54(−4)	4.79(−5)
		1.5	1.82(−4)	6.10(−5)	1.83(−4)	6.26(−5)	1.92(−4)	5.66(−5)
		2.0	2.17(−4)	7.13(−5)	2.16(−4)	7.32(−5)	2.34(−4)	6.72(−5)
16	16	0.5	6.30(−10)	2.37(−10)	6.75(−10)	2.76(−10)	1.23(−8)	3.63(−9)
		1.0	6.76(−10)	2.40(−10)	6.82(−10)	2.87(−10)	1.56(−8)	5.16(−9)
		1.5	8.00(−10)	2.65(−10)	7.57(−10)	3.18(−10)	1.99(−8)	6.76(−9)
		2.0	9.68(−10)	3.35(−10)	8.81(−10)	3.49(−10)	2.48(−8)	8.53(−9)
32	4	0.5	5.25(−4)	2.10(−4)	5.24(−4)	2.10(−4)	2.37(−3)	7.14(−4)
		1.0	7.40(−4)	2.88(−4)	7.41(−4)	2.90(−4)	2.96(−3)	1.00(−3)
		1.5	1.04(−3)	4.25(−4)	1.04(−3)	4.27(−4)	3.75(−3)	1.33(−3)
		2.0	1.40(−3)	5.87(−4)	1.40(−3)	5.89(−4)	4.62(−3)	1.67(−3)
	8	0.5	1.78(−6)	7.32(−7)	1.93(−6)	7.95(−7)	9.50(−7)	3.05(−7)
		1.0	2.41(−6)	1.06(−6)	2.62(−6)	1.15(−6)	1.24(−6)	5.22(−7)
		1.5	3.23(−6)	1.39(−6)	3.51(−6)	1.51(−6)	1.64(−6)	7.73(−7)
		2.0	4.17(−6)	1.79(−6)	4.52(−6)	1.92(−6)	2.09(−6)	1.03(−6)
	16	0.5	6.95(−11)	2.93(−11)	9.52(−11)	4.00(−11)	1.90(−10)	5.73(−11)
		1.0	7.48(−11)	3.23(−11)	1.03(−10)	4.42(−11)	2.40(−10)	8.15(−11)
		1.5	8.06(−11)	3.51(−11)	1.11(−10)	4.81(−11)	3.04(−10)	1.08(−10)
		2.0	8.67(−11)	3.79(−11)	1.19(−10)	5.19(−11)	3.76(−10)	1.36(−10)
	32	0.5	1.02 (−14)	6.99(−15)	1.36(−14)	1.21(−14)	1.10(−14)	7.88(−15)
		1.0	2.03(−14)	1.45(−14)	2.51(−14)	1.80(−14)	2.22(−14)	1.54(−14)
		1.5	2.98(−14)	1.88(−14)	3.68(−14)	2.74(−14)	3.54(−14)	2.22(−14)
		2.0	4.05(−14)	2.31(−14)	5.04(−14)	3.13(−14)	4.80(−14)	3.12(−14)

be easily achieved by selecting an M value for a given N. In particular, if $M \ll N$, the DSC approach behaves truly like a finite-difference scheme. To achieve optimal (or near optimal) accuracy, the σ is chosen in proportion to M and Δ. When $M = 32, 16, 8, 4, 2$ and 1, $\frac{\sigma}{\Delta}$ are chosen as 3.2, 2.5, 1.8, 1.2, 0.9 and 0.6 for both RSK and RDK, and 2.8, 2.0, 1.6 1.0 0.8 and 0.6 for RLK. We compute the L_2 and L_∞ errors of u for a number of combinations of N and M and the results are listed in table 4 for four different times ($t = 0.5, 1.0, 1.5, 2.0$). A good consistency in the accuracy among solutions at different times (or equivalently, over 2000 iterations) is observed. The DSC results are quite accurate when $N = M = 4$ and are of machine precision when $N = M = 32$. It is interesting to note that for fixed $M = 4$, the

results of $N = 32$ and 4 differ little in accuracy. We also checked the DSC finite-difference approximation at the tridiagonal matrix limit ($M = 1$) and the result is very good for $N = 4$ (i.e. a total of four interior points in a $(2\pi)^2$ box).

4. Discussion

The main purpose of this paper is to discuss the unified features of the discrete singular convolution algorithm [25]. It is found that the implementations of the DSC algorithm using a number of computational methods can be deduced from a single starting point, the method of weighted residuals. This chain of deduction provides a unified approach for solving the Fokker–Planck equation and other differential equations in general. Some of these deduction relations are novel to the best of our knowledge.

We demonstrate that by adjusting the support of the DSC trial functions, the DSC algorithm can be easily implemented either as a local method or as a global method. For this reason, the present DSC approach has the accuracy of the global method, while maintaining the flexibility of the local method for handling complex boundaries and geometry. In fact, the solution of the Fokker–Planck equation of a previous paper [25] was obtained by using the global limit. Whereas, in the present computations, a local approximation is used for all Fokker–Planck problems. A comparison between global and local DSC treatments is given in solving the Euler equation.

We also show that the DSC implementations of Galerkin and collocation are computationally equivalent, i.e. the collocation, equation (8) can be deduced from the Galerkin equation (6) because of the choice of the DSC trial functions. Galerkin methods have a profound influence on the theory of approximations. Both spectral methods and finite-element methods are often formulated within the framework of the Galerkin approach. There has been a great deal of argument about the advantages and disadvantages of the Galerkin approach in comparison to many other methods. The present DSC approach might provide a unified framework for a discussion of these methods.

The present Galerkin-induced collocation scheme provides a natural base for the realization of finite-difference methods. High-order finite difference is not a new idea in numerical approximations [31]. However, the mathematical construction of high-order finite-difference schemes often become too cumbersome to use in practical applications as the order increases. The present DSC approach provides a simple, systematic algorithm for the generation of finite-difference schemes of an arbitrary order. The implementation of this finite difference is demonstrated by solving the Euler equation with a number of different matrix bandwidths.

Recently, wavelet theory and techniques have had great success in signal processing, data compression and telecommunication. The two most important features of the wavelet theory are multiresolution analysis and time–frequency localization. Their potential applications in solving partial differential equations have been explored extensively [45–50] in the hope of finding unified approaches for numerical approximations. However, before wavelet approaches can be of practical use, a number of technical difficulties have to be overcome. In our view, the first difficulty is the implementation of boundary conditions in a multiresolution setting. The second difficulty is the requirement of sufficiently high wavelet regularity to provide sufficiently weak solutions. Moreover, there is a lack of general and systematic numerical algorithms for incorporating wavelets in an efficient manner. Nevertheless, the wavelet multiresolution analysis still has great potential for developing adaptive grid and multigrid algorithms. The present DSC algorithm is closely related to the wavelet theory [25, 35]. In fact, the DSC kernels

have a feature in common with wavelets in terms of time–frequency (position–momentum) localization. However, unlike in a wavelet algorithm, multiresolution analysis is feasible but it is not required in the DSC algorithm.

In contrast to our earlier work dealing with the application of the DSC approach to the Fokker–Planck equation via an eigenfunction expansion [25], we have explored in this paper a DSC-based time-propagation approach for solving the Fokker–Planck equation. Three typical DSC kernels, the regularized Shannon kernel, the regularized Dirichlet kernel and the regularized Lagrange kernel, are used as trial functions within the framework of the present method. Four benchmark examples are chosen to demonstrate the usefulness and to test the accuracy of the present DSC approach. The first example is the repulsive Wong process. This is used for objectively testing the ability of the method to handle the monomodality–bimodality transition. The Wong process requires a large computational domain to ensure that the boundary reflection of the density flux has little influence in a highly accurate computation. By using reasonable resolution, regularization and a quite large time increment, the present approach performs very well in characterizing the transition. In fact, the present unified approach delivers machine accuracy at an early time.

The Black–Scholes equation of option pricing was chosen as the second numerical example. This financial equation can be regarded as a reaction–diffusion equation, although, its derivation was entirely based on stochastic analysis. By using a simple transformation, the Black–Scholes equation is converted into the standard form of the Fokker–Planck equation which admits an analytical solution. The present numerical results for the Black–Scholes equation are obtained by using three different DSC kernels with a reasonable resolution and relatively large time mesh. The extremely high accuracy in the present calculation indicates that the DSC-based unified algorithm is a valuable potential approach for various option pricing simulations.

The third example treated is a nonlinear stochastic model. The effective potential of the corresponding Fokker–Planck equation is time dependent through the first-order moment of the transition probability density. Despite the nonlinearity and the poor approximation of the initial density distribution, the numerical solutions quickly settle to a smooth, stable and correct distribution after a few iterations. This is due to the fact that the DSC trial functions are chosen as Schwartz class functions and they are capable of auto-regularizing when used as integration kernels. Our results are of machine precision at a relatively later time. To the best of our knowledge, this is the best numerical solution to this nonlinear Fokker–Planck equation to date. These illustrative calculations indicate that the present unified approach is extremely accurate, efficient and robust for numerical simulations of stochastic systems.

A common feature in the above-mentioned Fokker–Planck equation is that the equation is of strong parabolic type and the solution decays as time increases. Therefore, it is necessary to employ an additional example to validate the present DSC algorithm further for handling more complicated partial differential equations. To this end, we choose the incompressible Euler equation where its velocity field equations are of strong hyperbolic type and a derived equation for the pressure is of elliptic type. A standard implicit Euler scheme is used for the time discretization and at each time t_{n+1}, linear algebraic equations are constructed by using the collocation method. In the present approach, carrying out differentiations in the collocation is equivalent to implementing the finite-difference weights computed from the DSC trial functions. We test the DSC algorithm by using 4, 8, 16 and 32 grid points in each dimension in association with many different half-matrix bandwidths ($M = 4, 8, 16$ and 32). As expected, the DSC algorithm achieves its highest accuracy at the global limit ($M = N$) for each given N. The machine precision is reached when $N = M = 32$. Very good results are also obtained for many banded matrix calculations. We believe that the feature of being able

to provide both global and local approximations in one formulation is of practical importance for large-scale computations.

Although this paper emphasizes the connection of a few computational methods and the unified features of the DSC approach, it claims neither that all computational methods are the same, nor that the DSC algorithm engulfs all methods. For example, it is still not clear whether the DSC algorithm is applicable in adaptive and unstructured grids (progress has been made on a DSC multigrid method). The reader is urged to keep the distinction of various methods in mind and maintain a perspective.

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